Metric adjusted skew information

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Abstract

We extend the concept of Wigner-Yanase-Dyson skew information to something we call "metric adjusted skew information" (of a state with respect to a conserved observable). This "skew information" is intended to be a non-negative quantity bounded by the variance (of an observable in a state) that vanishes for observables commuting with the state. We show that the skew information is a convex function on the manifold of states. It also satisfies other requirements, proposed by Wigner and Yanase, for an effective measure-of-information content of a state relative to a conserved observable. We establish a connection between the geometrical formulation of quantum statistics as proposed by Chentsov and Morozova and measures of quantum information as introduced by Wigner and Yanase and extended in this article. We show that the set of normalized Morozova-Chentsov functions describing the possible quantum statistics is a Bauer simplex and determine its extreme points. We determine a particularly simple skew information, the "λ-skew information," parametrized by a $\lambda \in (0, 1]$, and show that the convex cone this family generates coincides with the set of all metric adjusted skew informations.

Key words: Skew information, convexity, monotone metric, Morozova-Chentsov function, λ-skew information.

1 Introduction

In the mathematical model for a quantum mechanical system, the physical observables are represented by self-adjoint operators on a Hilbert space. The "states" (that is, the "expectation functionals" associated with the states) of the physical system are often "modeled" by the unit vectors in the underlying Hilbert space. So, if A represents an observable and $x \in H$ corresponds to

a state of the system, the expectation of A in that state is $(Ax \mid x)$. For what we shall be proving, it will suffice to assume that our Hilbert space is finite dimensional and that the observables are self-adjoint operators, or the matrices that represent them, on that finite dimensional space. In this case, the states can be realized with the aid of the trace (functional) on matrices and an associated "density matrix". We denote by $\mathrm{Tr}(B)$ the usual trace of a matrix B (that is, $\mathrm{Tr}(B)$ is the sum of the diagonal entries of B). The expectation functional of a state can be expressed as $\mathrm{Tr}(\rho A)$, where ρ is a matrix, the density matrix associated with the state, and "$\mathrm{Tr}(\rho A)$" is the trace of the product ρA of the two matrices ρ and A. (Henceforth, we write "$\mathrm{Tr}\,\rho A$" omitting the parentheses when they are clearly understood.)

In [21], Wigner noticed that the obtainable accuracy of the measurement of a physical observable represented by an operator that does not commute with a conserved quantity (observable) is limited by the "extent" of that non-commutativity. Wigner proved it in the simple case where the physical observable is the x-component of the spin of a spin one-half particle and the z-component of the angular momentum is conserved. Araki and Yanase [1] demonstrated that this is a general phenomenon and pointed out, following Wigner's example, that under fairly general conditions an approximate measurement may be carried out.

Another difference is that observables that commute with a conserved additive quantity, like the energy, components of the linear or angular momenta, or the electrical charge, can be measured easily and accurately by microscopic apparatuses (the analysis is restricted to one conserved quantity), while other observables can be only approximately measured by a macroscopic apparatus large enough to superpose sufficiently many states with different quantum numbers of the conserved quantity.

Wigner and Yanase [22] proposed finding a measure of our knowledge of a difficult-to-measure observable with respect to a conserved quantity. The quantum mechanical entropy is a measure of our ignorance of the state of a system, and minus the entropy can therefore be considered as an expression of our knowledge of the system. This measure has many attractive properties but does not take into account the conserved quantity. In particular, Wigner and Yanase wanted a measure that vanishes when the observable commutes with the conserved quantity. It should therefore not measure the effect of mixing in the classical sense as long as the pure states taking part in the mixing commute with the conserved quantity. Only transition probabilities of pure states "lying askew" (to borrow from the introduction of [22]) to the eigenvectors of the conserved quantity should give contributions to the proposed measure.

Wigner and Yanase discussed a number of requirements that such a mea-

sure should satisfy in order to be meaningful and suggested, tentatively, the skew information defined by

$$I(\rho, A) = -\tfrac{1}{2}\operatorname{Tr}\bigl([\rho^{1/2}, A]^2\bigr),$$

where $[C, D]$ is the usual "bracket notation" for operators or matrices: $[C, D] = CD - DC$, as a measure of the information contained in a state ρ with respect to a conserved observable A. It manifestly vanishes when ρ commutes with A, and it is homogeneous in ρ.

The requirements Wigner and Yanase discussed, all reflected properties considered attractive or even essential. Since information is lost when separated systems are united such a measure should be decreasing under the mixing of states, that is, be convex in ρ. The authors proved this for the skew information, but noted that other measures may enjoy the same properties; in particular, the expression

$$-\tfrac{1}{2}\operatorname{Tr}[\rho^p, A][\rho^{1-p}, A] \qquad 0 < p < 1$$

proposed by Dyson. Convexity of this expression in ρ became the celebrated Wigner-Yanase-Dyson conjecture which was later proved by Lieb [14]. (See also [7] for a truly elementary proof.)

The measure should also be additive with respect to the aggregation of isolated subsystems and, for an isolated system, independent of time. These requirements are discussed in more detail in section 3.1. They are easily seen to be satisfied by the skew information.

In the process that is the opposite of mixing, the information content should decrease. This requirement comes from thermodynamics where it is satisfied for both classical and quantum mechanical systems. It reflects the loss of information about statistical correlations between two subsystems when they are only considered separately. Wigner and Yanase conjectured that the skew information also possesses this property. They proved it when the state of the aggregated system is pure[1].

The aim of this article is to connect the subject of measures of quantum information as laid out by Wigner and Yanase with the geometrical formulation of quantum statistics by Chentsov, Morozova and Petz.

The Fisher information measures the statistical distinguishability of probability distributions. Let $\mathcal{P}_n = \{p = (p_1, \ldots, p_n) \mid p_i > 0\}$ be the (open) probability simplex with tangent space $T\mathcal{P}_n$. The Fisher-Rao metric is then given by

$$M_p(u, v) = \sum_{i=1}^{n} \frac{u_i v_i}{p_i} \qquad u, v \in T\mathcal{P}_n.$$

[1] We subsequently demonstrated [9] that the conjecture fails for general mixed states.

Note that $u = (u_1, \dots, u_n) \in T\mathcal{P}_n$ if and only if $u_1 + \cdots + u_n = 0$, but that the metric is well-defined also on $\mathbf{R}^n$. Chentsov proved that the Fisher-Rao metric is the unique Riemannian metric contracting under Markov morphisms [2].

Since Markov morphisms represent coarse graining or randomization, it means that the Fisher information is the only Riemannian metric possessing the attractive property that distinguishability of probability distributions becomes more difficult when they are observed through a noisy channel.

Chentsov and Morozova extended the analysis to quantum mechanics by replacing Riemannian metrics defined on the tangent space of the simplex of probability distributions with positive definite sesquilinear (originally bilinear) forms K_ρ defined on the tangent space of a quantum system, where ρ is a positive definite state. Customarily, K_ρ is extended to all operators (matrices) supported by the underlying Hilbert space, cf. [19, 6] for details. Noisy channels are in this setting represented by stochastic (completely positive and trace preserving) mappings T, and the contraction property by the monotonicity requirement

$$K_{T(\rho)}(T(A), T(A)) \le K_\rho(A, A)$$

is imposed for every stochastic mapping $T : M_n(\mathbf{C}) \to M_m(\mathbf{C})$. Unlike the classical situation, it turned out that this requirement no longer uniquely determines the metric. By the combined efforts of Chentsov, Morozova and Petz it is established that the monotone metrics are given on the form

$$K_\rho(A, B) = \operatorname{Tr} A^* c(L_\rho, R_\rho) B, \tag{1}$$

where c is a so called Morozova-Chentsov function and $c(L_\rho, R_\rho)$ is the function taken in the pair of commuting left and right multiplication operators (denoted L_ρ and R_ρ respectively) by ρ. The Morozova-Chentsov function is of the form

$$c(x, y) = \frac{1}{y f(xy^{-1})} \qquad x, y > 0,$$

where f is a positive operator monotone function defined in the positive half-axis satisfying the functional equation

$$f(t) = t f(t^{-1}) \qquad t > 0. \tag{2}$$

The function

$$f(t) = \frac{t + 2\sqrt{t} + 1}{4} \qquad t > 0$$

is clearly operator monotone and satisfies (2). The associated Morozova-Chentsov function

$$c^{WY}(x, y) = \frac{4}{(\sqrt{x} + \sqrt{y})^2} \qquad x, y > 0$$

therefore defines a monotone metric

$$K_\rho^{WY}(A,B) = \operatorname{Tr} A^* c^{WY}(L_\rho, R_\rho) B,$$

which we shall call the Wigner-Yanase metric. The starting point of our investigation is the observation by Gibilisco and Isola [4] that

$$I(\rho, A) = \tfrac{1}{8} \operatorname{Tr} i[\rho, A] c^{WY}(L_\rho, R_\rho) i[\rho, A].$$

There is thus a relationship between the Wigner-Yanase measure of quantum information and the geometrical theory of quantum statistics. It is the aim of the present article to explore this relationship in detail. The main result is that all well-behaved measures of quantum information - including the Wigner-Yanase-Dyson skew informations - are given in this way for a suitable subclass of monotone metrics.

1.1 Regular metrics

Definition 1.1 (Regular metric)**.** *We say that a symmetric monotone metric [16, 20] on the state space of a quantum system is regular, if the corresponding Morozova-Chentsov function c admits a strictly positive limit*

$$m(c) = \lim_{t \to 0} c(t,1)^{-1}.$$

We call $m(c)$ the metric constant.

We also say, more informally, that a Morozova-Chentsov function c is regular if $m(c) > 0$. The function $f(t) = c(t,1)^{-1}$ is positive and operator monotone on the positive half-line and may be extended to the closed positive half-line. Thus the metric constant $m(c) = f(0)$.

Definition 1.2 (metric adjusted skew information)**.**
Let c be the Morozova-Chentsov function of a regular metric. We introduce the metric adjusted skew information $I_\rho^c(A)$ by setting

$$\begin{aligned} I_\rho^c(A) &= \tfrac{m(c)}{2} K_\rho^c(i[\rho, A], i[\rho, A]) \\ &= \tfrac{m(c)}{2} \operatorname{Tr} i[\rho, A] c(L_\rho, R_\rho) i[\rho, A] \end{aligned} \tag{3}$$

for every $\rho \in \mathcal{M}_n$ (the manifold of states) and every self-adjoint $A \in M_n(\mathbf{C})$.

Note that the metric adjusted skew information is proportional to the square of the metric length, as it is calculated by the symmetric monotone

metric K_ρ^c with Morozova-Chentsov function c, of the commutator $i[\rho, A]$, and that this commutator belongs to the tangent space of the state manifold $\mathcal{M}_n$. Metric adjusted skew information is thus a non-negative quantity. If we consider the WYD-metric with Morozova-Chentsov function

$$c^{WYD}(x,y) = \frac{1}{p(1-p)} \cdot \frac{(x^p - y^p)(x^{1-p} - y^{1-p})}{(x-y)^2} \qquad 0 < p < 1,$$

then the metric constant $m(c^{WYD}) = p(1-p)$ and the metric adjusted skew information

$$\begin{aligned} I_\rho^{c^{WYD}}(A) &= \tfrac{p(1-p)}{2} \operatorname{Tr} i[\rho, A] c^{WYD}(L_\rho, R_\rho) i[\rho, A] \\ &= -\tfrac{1}{2} \operatorname{Tr}[\rho^p, A][\rho^{1-p}, A] \end{aligned}$$

becomes the Dyson generalization of the Wigner-Yanase skew information[2]. The choice of the factor $m(c)$ therefore works also for $p \neq 1/2$. It is in fact a quite general construction, and the metric constant is related to the topological properties of the metric adjusted skew information close to the border of the state manifold. But it is difficult to ascertain these properties directly, so we postpone further investigation until having established that $I_\rho^c(A)$ is a convex function in ρ. Since the commutator $i[\rho, A] = i(L_\rho - R_\rho)A$ we may rewrite the metric adjusted skew information as

$$\begin{aligned} I_\rho^c(A) &= \tfrac{m(c)}{2} \operatorname{Tr} A (i(L_\rho - R_\rho))^* c(L_\rho, R_\rho) i(L_\rho - R_\rho) A \\ &= \tfrac{m(c)}{2} \operatorname{Tr} A\, \hat{c}(L_\rho, R_\rho) A, \end{aligned} \tag{4}$$

where

$$\hat{c}(x,y) = (x-y)^2 c(x,y) \qquad x, y > 0. \tag{5}$$

Before we can address these questions in more detail, we have to study various characterizations of (symmetric) monotone metrics.

2 Characterizations of monotone metrics

Theorem 2.1. *A positive operator monotone decreasing function g defined in the positive half-axis and satisfying the functional equation*

$$g(t^{-1}) = t \cdot g(t) \tag{6}$$

[2]Hasegawa and Petz proved in [18] that the function c^{WYD} is a Morozova-Chentsov function. They also proved that the Wigner-Yanase-Dyson skew information is proportional to the (corresponding) quantum Fisher information of the commutator $i[\rho, A]$.

has a canonical representation

$$g(t)=\int_0^1\left(\frac{1}{t+\lambda}+\frac{1}{1+t\lambda}\right)d\mu(\lambda), \tag{7}$$

where μ is a finite Borel measure with support in $[0,1]$.

Proof. The function g is necessarily of the form

$$g(t)=\beta+\int_0^\infty\frac{1}{t+\lambda}\,d\mu(\lambda),$$

where $\beta\ge 0$ is a constant and μ is a positive Borel measure such that the integrals $\int(1+\lambda^2)^{-1}d\mu(\lambda)$ and $\int\lambda(1+\lambda^2)^{-1}d\mu(\lambda)$ are finite, cf. [8, Page 9]. We denote by $\tilde\mu$ the measure obtained from μ by removing a possible atom in zero. Then, by making the transformation $\lambda\to\lambda^{-1}$, we may write

$$\begin{aligned}g(t)&=\beta+\frac{\mu(0)}{t}+\int_0^\infty\frac{1}{t+\lambda}\,d\tilde\mu(\lambda)\\&=\beta+\frac{\mu(0)}{t}+\int_0^\infty\frac{1}{t+\lambda^{-1}}\cdot\frac{1}{\lambda^2}\,d\tilde\mu(\lambda^{-1})\\&=\beta+\frac{\mu(0)}{t}+\int_0^\infty\frac{1}{1+t\lambda}\,d\nu(\lambda),\end{aligned}$$

where ν is the Borel measure given by $d\nu(\lambda)=\lambda^{-1}d\tilde\mu(\lambda^{-1})$. Since g satisfies the functional equation (6) we obtain

$$\beta+\mu(0)t+\int_0^\infty\frac{1}{1+t^{-1}\lambda}\,d\nu(\lambda)=t\beta+\mu(0)+\int_0^\infty\frac{t}{t+\lambda}\,d\tilde\mu(\lambda).$$

By letting $t\to 0$ and since ν and $\tilde\mu$ have no atoms in zero, we obtain $\beta=\mu(0)$ and consequently

$$\int_0^\infty\frac{1}{t+\lambda}\,d\nu(\lambda)=\int_0^\infty\frac{1}{t+\lambda}\,d\tilde\mu(\lambda)\qquad t>0.$$

By analytic continuation we realize that both measures ν and $\tilde\mu$ appear as the representing measure of an analytic function with negative imaginary part in the complex upper half plane. They are therefore, by the representation

theorem for this class of functions, necessarily identical. We finally obtain

$$\begin{aligned} g(t) &= \beta + \frac{\beta}{t} + \int_0^\infty \frac{1}{t+\lambda}\, d\tilde{\mu}(\lambda) \\ &= \beta + \frac{\beta}{t} + \int_0^1 \frac{1}{t+\lambda}\, d\tilde{\mu}(\lambda) + \int_0^1 \frac{1}{t+\lambda^{-1}} \cdot \frac{1}{\lambda^2}\, d\tilde{\mu}(\lambda^{-1}) \\ &= \beta + \frac{\beta}{t} + \int_0^1 \frac{1}{t+\lambda}\, d\tilde{\mu}(\lambda) + \int_0^1 \frac{1}{1+t\lambda}\, d\nu(\lambda) \\ &= \beta + \frac{\beta}{t} + \int_0^1 \left(\frac{1}{t+\lambda} + \frac{1}{1+t\lambda}\right) d\tilde{\mu}(\lambda) \\ &= \int_0^1 \left(\frac{1}{t+\lambda} + \frac{1}{1+t\lambda}\right) d\mu(\lambda). \end{aligned}$$

The statement follows since every function of this form obviously is operator monotone decreasing and satisfy the functional equation (6). We also realize that the representing measure μ is uniquely defined. **QED**

Remark 2.2. Inspection of the proof of Theorem 2.1 shows that the Pick function $-g(x) = -c(x,1)$ has the canonical representation

$$-g(x) = -g(0) + \int_{-\infty}^0 \frac{1}{\lambda - t}\, d\mu(-\lambda).$$

The representing measure therefore appears as $1/\pi$ times the limit measure of the imaginary part of the analytic continuation $-g(z)$ as z approaches the closed negative half-axis from above, cf. for example [3]. The measure μ in (7) therefore appears as the image of the representing measure's restriction to the interval $[-1, 0]$ under the transformation $\lambda \to -\lambda$.

We define, in the above setting, an equivalent Borel measure μ_g on the closed interval $[0, 1]$ by setting

$$d\mu_g(\lambda) = \frac{2}{1+\lambda}\, d\mu(\lambda) \tag{8}$$

and obtain:

Corollary 2.3. *A positive operator monotone decreasing function g defined in the positive half-axis and satisfying the functional equation (6) has a canonical representation*

$$g(t) = \int_0^1 \frac{1+\lambda}{2} \left(\frac{1}{t+\lambda} + \frac{1}{1+t\lambda}\right) d\mu_g(\lambda), \tag{9}$$

where μ_g is a finite Borel measure with support in $[0,1]$. The function g is normalized in the sense that $g(1)=1$, if and only if μ_g is a probability measure.

Corollary 2.4. *A Morozova-Chentsov function c allows a canonical representation of the form*

$$c(x,y)=\int_0^1 c_\lambda(x,y)\,d\mu_c(\lambda)\qquad x,y>0, \tag{10}$$

where μ_c is a finite Borel measure on $[0,1]$ and

$$c_\lambda(x,y)=\frac{1+\lambda}{2}\left(\frac{1}{x+\lambda y}+\frac{1}{\lambda x+y}\right)\qquad \lambda\in[0,1]. \tag{11}$$

The Morozova-Chentsov function c is normalized in the sense that $c(1,1)=1$ (corresponding to a Fisher adjusted metric), if and only if μ_c is a probability measure.

Proof. A Morozova-Chentsov function is of the form $c(x,y)=y^{-1}f(xy^{-1})^{-1}$, where f is a positive operator monotone function defined in the positive half-axis and satisfying the functional equation $f(t)=tf(t^{-1})$. The function $g(t)=f(t)^{-1}$ is therefore operator monotone decreasing and satisfies the functional equation (6). It is consequently of the form (9) for some finite Borel measure μ_g. Since also $c(x,y)=y^{-1}g(xy^{-1})$ the assertion follows by setting $\mu_c=\mu_g$. **QED**

We have shown that the set of normalized Morozova-Chentsov functions is a Bauer simplex, and that the extreme points exactly are the functions of the form (11).

Theorem 2.5. *We exhibit the measure μ_c in the canonical representation (10) for a number of Morozova-Chentsov functions.*

1. *The Wigner-Yanase-Dyson metric with (normalized) Morozova-Chentsov function*
$$c(x,y)=\frac{1}{p(1-p)}\cdot\frac{(x^p-y^p)(x^{1-p}-y^{1-p})}{(x-y)^2}$$
is represented by
$$d\mu_c(\lambda)=\frac{2\sin p\pi}{\pi p(1-p)}\cdot\frac{\lambda^p+\lambda^{1-p}}{(1+\lambda)^3}\,d\lambda$$
for $0<p<1$.

The Wigner-Yanase metric is obtained by setting $p = 1/2$ and it is represented by

$$d\mu_c(\lambda) = \frac{16\lambda^{1/2}}{\pi(1+\lambda)^3}\, d\lambda.$$

2. *The Kubo metric with (normalized) Morozova-Chentsov function*

$$c(x,y) = \frac{\log x - \log y}{x-y}$$

is represented by

$$d\mu_c(\lambda) = \frac{2}{(1+\lambda)^2}\, d\lambda.$$

3. *The increasing bridge with (normalized) Morozova-Chentsov functions*

$$c_\gamma(x,y) = x^{-\gamma} y^{-\gamma} \left(\frac{x+y}{2}\right)^{2\gamma-1}$$

is represented by

$$\begin{cases} \mu_c &= \delta(\lambda - 1) & \gamma = 0 \\ d\mu_c(\lambda) &= \dfrac{2\sin\gamma\pi}{(1+\lambda)\pi}\lambda^{-\gamma}\left(\dfrac{1-\lambda}{2}\right)^{2\gamma-1} d\lambda & 0 < \gamma < 1 \\ \mu_c &= \delta(\lambda) & \gamma = 1, \end{cases}$$

where δ is the Dirac measure with unit mass in zero.

Proof. We calculate the measures by the method outlined in Remark 2.2.

1. For the Wigner-Yanase-Dyson metric we therefore consider the analytic continuation

$$-g(re^{i\phi}) = -c(re^{i\phi}, 1) = \frac{-1}{p(1-p)} \cdot \frac{(r^p e^{ip\phi} - 1)(r^{1-p} e^{i(1-p)\phi} - 1)}{(re^{i\phi} - 1)^2}$$

where $r > 0$ and $0 < \phi < \pi$. We calculate the imaginary part and note that $r \to -\lambda$ and $\phi \to \pi$ for $z \to \lambda < 0$. We make sure that the representing measure has no atom in zero and obtain the desired expression by tedious but elementary calculations.

2. For the Kubo metric we consider

$$-g(x) = -c(x,1) = -\frac{\log x}{x-1}$$

and calculate the imaginary part

$$-\Im g(re^{i\phi}) = \frac{2r\log r \sin\phi + \phi - \phi r\cos\phi}{r^2 - 2r\cos\phi + 1}$$

of the analytic continuation. It converges towards $\pi/(1-\lambda)$ for $z \to \lambda < 0$ and towards $\pi/2$ for $z = re^{i\pi} \to 0$. The representing measure has therefore no atom in zero, and $d\mu(\lambda) = d\lambda/(1+\lambda)$ which may be verified by direct calculation.

3. For the increasing bridge we consider

$$-g_\gamma(x) = -c_\gamma(x,1) = -x^{-\gamma}\left(\frac{x+1}{2}\right)^{2\gamma-1}$$

and calculate the imaginary part

$$-\Im g_\gamma(re^{i\phi}) = -r^{-\gamma} r_1^{2\gamma-1} \exp i(-\gamma\phi + (2\gamma-1)\theta)$$

of the analytic continuation, where

$$r_1 = \tfrac{1}{2}(r^2 + 2r\cos\phi + 1)^{1/2} \quad \text{and} \quad \theta = \arctan\frac{r sin\phi}{1 + r\cos\phi}.$$

We first note that $\theta = \pi/2$ and $r_1 = (r\sin\phi)/2$ for $\lambda = -1$, and that $\theta \to 0$ and $r_1 \to (1+\lambda)/2$ for $-1 < \lambda \leq 0$. The statement now follows by examination of the different cases. **QED**

In the reference [6] we proved the following exponential representation of the Morozova-Chentsov functions.

Theorem 2.6. *A Morozova-Chentsov function c admits a canonical representation*

$$c(x,y) = \frac{C_0}{x+y}\exp\int_0^1 \frac{1-\lambda^2}{\lambda^2+1}\cdot\frac{x^2+y^2}{(x+\lambda y)(\lambda x+y)}h(\lambda)\,d\lambda \tag{12}$$

where $h:[0,1]\to[0,1]$ is a measurable function and C_0 is a positive constant. Both C_0 and the equivalence class containing h are uniquely determined by c. Any function c on the given form is a Morozova-Chentsov function.

Theorem 2.7. *We exhibit the constant C_0 and the representing function h in the canonical representation (12) for a number of Morozova-Chentsov functions.*

1. *The Wigner-Yanase-Dyson metric with Morozova-Chentsov function*

$$c(x,y) = \frac{1}{p(1-p)} \cdot \frac{(x^p - y^p)(x^{1-p} - y^{1-p})}{(x-y)^2}$$

is represented by

$$C_0 = \frac{\sqrt{2}}{p(1-p)} \left(1 - \cos p\frac{\pi}{2}\right)^{1/2} \left(1 - \cos(1-p)\frac{\pi}{2}\right)^{1/2}$$

and

$$h(\lambda) = \frac{1}{\pi} \arctan \frac{(\lambda^p + \lambda^{1-p}) \sin p\pi}{1 - \lambda - (\lambda^p - \lambda^{1-p}) \cos p\pi} \qquad 0 < \lambda < 1,$$

for $0 < p < 1$. *Note that* $0 \le h \le 1/2$.

The Wigner-Yanase metric is obtained by setting $p = 1/2$ *and is represented by*

$$C_0 = 4(\sqrt{2} - 1)$$

and

$$h(\lambda) = \frac{1}{\pi} \arctan \frac{2\lambda^{1/2}}{1 - \lambda} \qquad 0 < \lambda < 1.$$

2. *The Kubo metric with Morozova-Chentsov function*

$$c(x,y) = \frac{\log x - \log y}{x - y}$$

is represented by

$$C_0 = \frac{\pi}{2} \quad \textit{and} \quad h(\lambda) = \frac{1}{2} - \frac{1}{\pi} \arctan\left(-\frac{\log \lambda}{\pi}\right).$$

Note that $0 \le h \le 1/2$.

3. *The increasing bridge with Morozova-Chentsov functions*

$$c_\gamma(x,y) = x^{-\gamma} y^{-\gamma} \left(\frac{x+y}{2}\right)^{2j-1}$$

is represented by

$$C_0 = 2^{1-\gamma} \quad \textit{and} \quad h(\lambda) = \gamma, \qquad 0 \le \gamma \le 1.$$

Setting $\gamma = 0$, *we obtain that the Bures metric with Morozova-Chentsov function* $c(x,y) = 2/(x+y)$ *is represented by* $C_0 = 2$ *and* $h(\lambda) = 0$.

Proof. The analytic continuation of the operator monotone function $g(x) = \log f(x)$ into the upper complex plane, where $f(x) = c(x,1)^{-1}$ is the operator monotone function representing [19] the Morozova-Chentsov function, has bounded imaginary part. The representing measure of the Pick function g is therefore absolutely continuous with respect to Lebesgue measure. Since f satisfies the functional equation $f(t) = tf(t^{-1})$ we only need to consider the restriction of the measure to the interval $[-1,0]$, and the function h appears [6] as the image under the transformation $\lambda \to -\lambda$ of the Radon-Nikodym derivative. In the same reference it is shown that the constant $C_0 = \sqrt{2}e^{-\beta}$ where $\beta = \Re \log f(i)$.

1. For the Wigner-Yanase-Dyson metric the corresponding operator monotone function

$$f(x) = \frac{1}{c(x,1)} = p(1-p)\frac{(x-1)^2}{(x^p-1)(x^{1-p}-1)}$$

and we calculate by tedious but elementary calculations

$$\lim_{z\to\lambda} \Im \log f(z) = -\frac{1}{2i}\log H \qquad \lambda \in (-1,0),$$

where

$$H = \frac{N}{((-\lambda)^{2p} - 2(-\lambda)^p \cos p\pi + 1)((-\lambda)^{2(1-p)} - 2(-\lambda)^{(1-p)}\cos(1-p)\pi + 1)}$$

and

$$\begin{aligned} N &= (-\lambda)^2 + 2(-\lambda)^{1+p}e^{ip\pi} + (-\lambda)^{2p}e^{2ip\pi} - 2(-\lambda)^{2-p}e^{-ip\pi} \\ &+ 4\lambda - 2(-\lambda)^p e^{ip\pi} + (-\lambda)^{2(1-p)}e^{-2ip\pi} + 2(-\lambda)^{1-p}e^{-ip\pi} + 1 \end{aligned}$$

happens to be the square of the complex number

$$(1+\lambda) - ((-\lambda)^p - (-\lambda)^{1-p})\cos p\pi - i((-\lambda)^p + (-\lambda)^{1-p})\sin p\pi$$

with positive real part and negative imaginary part. Since H has modulus one we can therefore write

$$H = e^{-2i\theta} \qquad \lambda \in (-1,0),$$

where $0 < \theta < \pi/2$ and

$$\tan\theta = \frac{((-\lambda)^p + (-\lambda)^{1-p})\sin p\pi}{1 + \lambda - ((-\lambda)^p - (-\lambda)^{1-p})\cos p\pi}$$

which implies the expression for h. The constant C_0 is obtained by a simple calculation.

2. For the Kubo metric the corresponding operator monotone function

$$f(x) = \frac{1}{c(x,1)} = \frac{x-1}{\log x}$$

and we obtain by setting $z = re^{i\phi}$ and $z - 1 = r_1 e^{i\phi_1}$ the expression

$$\Im \log f(z) = \frac{1}{2i}\left(\log\frac{\log r - i\phi}{\log r + i\phi} + 2i\phi_1\right) \qquad 0 < \phi < \phi_1 < \pi.$$

Since

$$\log\frac{\log r - i\phi}{\log r + i\phi} \to \log\frac{\log(-\lambda) - i\pi}{\log(-\lambda) + i\pi}$$

for $z \to \lambda \in (-1, 0)$ and

$$\frac{\log(-\lambda) - i\pi}{\log(-\lambda) + i\pi} = e^{-2i\theta} \quad \text{where} \quad \tan\theta = \frac{\pi}{\log(-\lambda)}$$

we obtain

$$\lim_{z\to\lambda} \Im \log f(z) = \pi - \theta \qquad \frac{\pi}{2} < \theta < \pi,$$

and thus

$$h(\lambda) = 1 - \frac{1}{\pi}\arctan\frac{\pi}{\log\lambda}.$$

The constant C_0 is obtained by a straightforward calculation.

3. The statement for the increasing bridge was proved in [6]. **QED**

3 Convexity statements

Proposition 3.1. *Every Morozova-Chentsov function c is operator convex, and the mappings*

$$(\rho, \delta) \to \operatorname{Tr} A^* c(L_\rho, R_\delta) A$$

and

$$\rho \to K_\rho^c(A, A)$$

defined on the state manifold are convex for arbitrary $A \in M_n(\mathbf{C})$.

Proof. Let c be a Morozova-Chentsov function. Since inversion is operator convex, it follows from the representation given in (10) that c as a function of two variables is operator convex. The two assertions now follows from [7, Theorem 1.1]. **QED**

Lemma 3.2. *Let $\lambda \geq 0$ be a constant. The functions of two variables*

$$f(t,s) = \frac{t^2}{t+\lambda s} \quad \textit{and} \quad g(t,s) = \frac{ts}{t+\lambda s}$$

are operator convex respectively operator concave on $(0,\infty)\times(0,\infty)$.

Proof. The first statement is an application of the convexity, due to Lieb and Ruskai, of the mapping $(A,B)\to AB^{-1}A$. Indeed, setting

$$C_1 = A_1\otimes I_2 + \lambda I_1\otimes B_1 \quad \text{and} \quad C_2 = A_2\otimes I_2 + \lambda I_1\otimes B_2$$

we obtain

$$\begin{aligned}
&f(tA_1+(1-t)A_2, tB_1+(1-t)B_2)\\
&= \Big((tA_1+(1-t)A_2)\otimes I_2\Big)(tC_1+(1-t)C_2)^{-1}\Big((tA_1+(1-t)A_2)\otimes I_2\Big)\\
&\leq t(A_1\otimes I_2)C_1^{-1}(A_1\otimes I_2)+(1-t)(A_2\otimes I_2)C_2^{-1}(A_2\otimes I_2)\\
&= tf(A_1,B_1)+(1-t)f(A_2,B_2) \qquad t\in[0,1].
\end{aligned}$$

The second statement is a consequence of the concavity of the harmonic mean

$$H(A,B) = 2(A^{-1}+B^{-1})^{-1}.$$

Indeed, we may assume $\lambda > 0$ and obtain

$$\begin{aligned}
&g(tA_1+(1-t)A_2, tB_1+(1-t)B_2)\\
&= \frac{1}{2}H\Big(t(\lambda^{-1}A_1\otimes I_2)+(1-t)(\lambda^{-1}A_2\otimes I_2), t(I_1\otimes B_1)+(1-t)(I_1\otimes B_2)\Big)\\
&\geq t\frac{1}{2}H(\lambda^{-1}A_1\otimes I_2, I_1\otimes B_1)+(1-t)\frac{1}{2}H(\lambda^{-1}A_2\otimes I_2, I_1\otimes B_2)\\
&= tg(A_1,B_1)+(1-t)g(A_2,B_2)
\end{aligned}$$

for $t\in(0,1]$. **QED**

Proposition 3.3. *Let c be a Morozova-Chentsov function. The function of two variables*

$$\hat{c}(x,y) = (x-y)^2c(x,y) \qquad x,y>0$$

is operator convex.

Proof. A Morozova-Chentsov function c allows the representation (10) where μ is some finite Borel measure with support in $[0,1]$. Since

$$\frac{(x-y)^2}{x+\lambda y}=\frac{x^2+y^2-2xy}{x+\lambda y}$$

by Lemma 3.2 is a sum of operator convex functions the assertion follows. **QED**

Proposition 3.4. *Let c be a regular Morozova-Chentson function. We may write $\hat{c}(x,y)=(x-y)^2c(x,y)$ on the form*

$$\hat{c}(x,y)=\frac{x+y}{m(c)}-d_c(x,y)\qquad x,y>0, \tag{13}$$

where the positive symmetric function

$$d_c(x,y)=\int_0^1 xy\cdot c_\lambda(x,y)\frac{(1+\lambda)^2}{\lambda}\,d\mu_c(\lambda) \tag{14}$$

is operator concave in the first quadrant, and the finite Borel measure μ_c is the representing measure in (10) of the Morozova-Chentsov function c. In addition, we obtain the expression

$$\begin{aligned} I^c_\rho(A) &= \tfrac{m(c)}{2}\operatorname{Tr} A\hat{c}(L_\rho,R_\rho)A \\ &= \operatorname{Tr}\rho A^2-\tfrac{m(c)}{2}\operatorname{Tr} A\,d_c(L_\rho,R_\rho)A \end{aligned} \tag{15}$$

for the metric adjusted skew information.

Proof. We first notice that

$$\int_0^1\frac{(1+\lambda)^2}{2\lambda}\,d\mu_c(\lambda)=\lim_{t\to 0}c(t,1)=\frac{1}{m(c)} \tag{16}$$

and obtain

$$\begin{aligned} d_c(x,y) &= \frac{x+y}{m(c)}-\hat{c}(x,y)\\ &= \frac{x+y}{m(c)}-(x-y)^2c(x,y)\\ &= (x+y)\int_0^1\frac{(1+\lambda)^2}{2\lambda}\,d\mu_c(\lambda)-(x-y)^2\int_0^1 c_\lambda(x,y)\,d\mu_c(\lambda)\\ &= \int_0^1\left((x+y)\frac{(1+\lambda)^2}{2\lambda}-(x-y)^2c_\lambda(x,y)\right)\,d\mu_c(\lambda). \end{aligned}$$

The asserted expression of d_c then follows by a simple calculation and the definition of $c_\lambda(x,y)$ as given in (11). The function d_c is operator concave in the first quadrant by Proposition 3.3. **QED**

Definition 3.5. *We call the function d_c defined in (14) the representing function for the metric adjusted skew information $I^c_\rho(A)$ with (regular) Morozova-Chentsov function c.*

We introduce for $0 < \lambda \le 1$ the λ-skew information $I_\lambda(\rho, A)$ by setting

$$I_\lambda(\rho, A) = I^{c_\lambda}_\rho(A).$$

The metric is regular with metric constant $m(c_\lambda) = 2\lambda(1+\lambda)^{-2}$ and the representing measure μ_{c_λ} is the Dirac measure in λ. The representing function for the metric adjusted skew information is thus given by

$$d_{c_\lambda}(x,y) = xy \cdot c_\lambda(x,y)\frac{(1+\lambda)^2}{\lambda} = \frac{m(c_\lambda)}{2}\, xy \cdot c_\lambda(x,y).$$

If we set

$$f_\lambda(x,y) = xy \cdot c_\lambda(x,y) = \frac{1+\lambda}{2}\left(\frac{xy}{x+\lambda y} + \frac{xy}{\lambda x + y}\right) \qquad x, y > 0, \tag{17}$$

we therefore obtain the expression

$$I_\lambda(\rho, A) = \operatorname{Tr} \rho A^2 - \operatorname{Tr} A f_\lambda(L_\rho, R_\rho) A \tag{18}$$

for the λ-skew information.

Corollary 3.6. *Let c be a regular Morozova-Chentsov function. The metric adjusted skew information may be written on the form*

$$I^c_\rho(A) = \tfrac{m(c)}{2} \int_0^1 I_\lambda(\rho, A)\frac{(1+\lambda)^2}{\lambda}\, d\mu_c(\lambda),$$

where μ_c is the representing measure and $m(c)$ is the metric constant.

Proof. By applying the expressions in (15) and (14) together with the observation in (16) we obtain

$$\begin{aligned} I^c_\rho(A) &= \operatorname{Tr} \rho A^2 - \tfrac{m(c)}{2} \int_0^1 \operatorname{Tr} A f_\lambda(L_\rho, R_\rho) A \frac{(1+\lambda)^2}{\lambda}\, d\mu_c(\lambda) \\ &= \tfrac{m(c)}{2} (\operatorname{Tr} \rho A^2 - \operatorname{Tr} A f_\lambda(L_\rho, R_\rho) A) \frac{(1+\lambda)^2}{\lambda}\, d\mu_c(\lambda) \end{aligned}$$

and the assertion follows. **QED**

3.1 Measures of quantum information

The next result is a direct generalization of the Wigner-Yanase-Dyson-Lieb convexity theorem.

Theorem 3.7. *Let c be a regular Morozova-Chentsov function. The metric adjusted skew information is a convex function, $\rho \to I^c_\rho(A)$, on the manifold of states for any self-adjoint $A \in M_n(\mathbf{C})$.*

Proof. The function $\hat{c}(x,y) = (x-y)^2 c(x,y)$ is by Proposition 3.3 operator convex. Applying the representation of the metric adjusted skew information given in (4), the assertion now follows from [7, Theorem 1.1]. **QED**

The above proof is particularly transparent for the Wigner-Yanase-Dyson metric, since the function

$$\begin{aligned} \hat{c}^{WYD}(\lambda,\mu) &= \frac{1}{p(1-p)}(\lambda^p - \mu^p)(\lambda^{1-p} - \mu^{1-p}) \\ &= \frac{1}{p(1-p)}(2 - \lambda^p \mu^{1-p} - \lambda^{1-p}\mu^p) \end{aligned}$$

is operator convex by the simple argument given in [7, Corollary 2.2].

Wigner and Yanase [22] discussed a number of other conditions which a good measure of the quantum information contained in a state with respect to a conserved observable should satisfy, but noted that convexity was the most obvious but also the most restrictive and difficult condition. In addition to the convexity requirement an information measure should be additive with respect to the aggregation of isolated systems. Since the state of the aggregated system is represented by $\rho = \rho_1 \otimes \rho_2$ where ρ_1 and ρ_2 are the states of the systems to be united, and the conserved quantity $A = A_1 \otimes 1 + 1 \otimes A_2$ is additive in its components, we obtain

$$[\rho, A] = [\rho_1, A_1] \otimes \rho_2 + \rho_1 \otimes [\rho_2, A_2].$$

Inserting ρ and A, as above, in the definition of the metric adjusted skew information (3), we obtain

$$\begin{aligned} I^c_\rho(A) &= \tfrac{m(c)}{2} \operatorname{Tr}\Big(i[\rho_1, A_1] \otimes \rho_2 + \rho_1 \otimes i[\rho_2, A_2] \Big) \\ &\qquad c(L_{\rho_1}, R_{\rho_1}) \otimes c(L_{\rho_2}, R_{\rho_2}) \Big(i[\rho_1, A_1] \otimes \rho_2 + \rho_1 \otimes i[\rho_2, A_2] \Big). \end{aligned}$$

The cross terms vanish because of the cyclicity of the trace, and since ρ_1 and ρ_2 have unit trace we obtain

$$I^c_\rho(A) = I^c_{\rho_1}(A_1) + I^c_{\rho_2}(A_2)$$

as desired. The metric adjusted skew information for an isolated system should also be independent of time. But a conserved quantity A in an isolated system commutes with the Hamiltonian H, and since the time evolution of ρ is given by $\rho_t = e^{itH}\rho e^{-itH}$ we readily obtain

$$I^c_{\rho_t}(A) = I^c_\rho(A) \qquad t \ge 0$$

by using the unitary invariance of the metric adjusted skew information.

The variance $\operatorname{Var}_\rho(A)$ of a conserved observable A with respect to a state ρ is defined by setting

$$\operatorname{Var}_\rho(A) = \operatorname{Tr} \rho A^2 - (\operatorname{Tr} \rho A)^2.$$

It is a concave function in ρ.

Theorem 3.8. *Let c be a regular Morozova-Chentsov function. The metric adjusted skew information $I^c(\rho, A)$ may for each conserved (self-adjoint) variable A be extended from the state manifold to the state space. Furthermore,*

$$I^c_\rho(A) = \operatorname{Var}_\rho(A)$$

if ρ is a pure state, and

$$0 \le I^c_\rho(A) \le \operatorname{Var}_\rho(A)$$

for any density matrix ρ.

Proof. We note that the representing function d in (14) may be extended to a continuous operator concave function defined in the closed first quadrant with $d(t,0) = d(0,t) = 0$ for every $t \ge 0$, and that $d(1,1) = 2/m(c)$. Since a pure state is a one-dimensional projection P, it follows from the representation in (4) and the formula (13) that

$$\begin{aligned}
I^c_P(A) &= \tfrac{m(c)}{2} \operatorname{Tr}\left(\frac{APA + AAP}{m(c)} - d(1,1)APAP\right)\\
&= \operatorname{Tr} PA^2 - \operatorname{Tr}(PAP)^2\\
&= \operatorname{Tr} PA^2 - (\operatorname{Tr} PA)^2\\
&= \operatorname{Var}_P(A).
\end{aligned}$$

An arbitrary state ρ is by the spectral theorem a convex combination $\rho = \sum_i \lambda_i P_i$ of pure states. Hence

$$I^c_\rho(A) \le \sum_i \lambda_i I^c_{P_i}(A) = \sum_i \lambda_i \operatorname{Var}_{P_i}(A) \le \operatorname{Var}_\rho(A),$$

where we used the convexity of the metric adjusted skew information and the concavity of the variance. **QED**

3.2 The metric adjusted correlation

We have developed the notion of metric adjusted skew information, which is a generalization of the Wigner-Yanase-Dyson skew information. It is defined for all regular metrics (symmetric and monotone), where the term regular means that the associated Morozova-Chentsov functions have continuous extensions to the closed first quadrant with finite values everywhere except in the point $(0,0)$.

Definition 3.9. *Let c be a regular Morozova-Chentsov function, and let d be the representing function (14). The metric adjusted correlation is defined by*

$$\mathrm{Corr}^c_\rho(A,B) = \mathrm{Tr}\,\rho A^* B - \tfrac{m(c)}{2}\,\mathrm{Tr}\,A^*\, d(L_\rho, R_\rho) B$$

for arbitrary matrices A and B.

Since d is symmetric, the metric adjusted correlation is a symmetric sesqui-linear form which by (15) satisfies

$$\mathrm{Corr}^c_\rho(A,A) = I^c_\rho(A) \qquad \text{for self-adjoint } A.$$

The metric adjusted correlation is not a real form on self-adjoint matrices, and it is not positive on arbitrary matrices. Therefore, Cauchy-Schwartz inequality only gives a bound

$$|\Re\,\mathrm{Corr}^c_\rho(A,B)| \le I^c_\rho(A)^{1/2} I^c_\rho(A)^{1/2} \le \mathrm{Var}_\rho(A)^{1/2}\cdot\mathrm{Var}_\rho(B)^{1/2} \tag{19}$$

for the real part of the metric adjusted correlation. However, since

$$\mathrm{Corr}^c_\rho(A,B) - \mathrm{Corr}^c_\rho(B,A) = \mathrm{Tr}\,\rho[A,B] \qquad A^*=A,\ B^*=B,$$

we obtain

$$\frac{1}{2}|\,\mathrm{Tr}\,\rho[A,B]| = |\Im\,\mathrm{Corr}^c_\rho(A,B)|$$

for self-adjoint A and B. The estimate in (19) can therefore not be used to improve Heisenberg's uncertainty relations[3].

[3] In the first version of this paper, which appeared on July 22 2006, the estimation in (19) was erroneously extended to the metric adjusted skew information itself and not only to the real part, cf. also Luo [15] and Kosaki [12]. The author is indebted to Gibilisco and Isola for pointing out the mistake.

3.3 The variant bridge

The notion of a regular metric seems to be very important. We note that the Wigner-Yanase-Dyson metrics and the Bures metric are regular, while the Kubo metric and the maximal symmetric monotone metric are not.

The continuously increasing bridge with Morozova-Chentsov functions

$$c_\gamma(x,y) = x^{-\gamma}y^{-\gamma}\left(\frac{x+y}{2}\right)^{2j-1} \qquad 0 \le \gamma \le 1$$

connects the Bures metric $c_0(x,y) = 2/(x+y)$ with the maximal symmetric monotone metric $c_1(x,y) = 2xy/(x+y)$. Since the Bures metric is regular and the maximal symmetric monotone metric is not, any bridge connecting them must fail to be regular at some point. However, the above bridge fails to be regular at any point $\gamma \neq 0$. A look at the formula (12) shows that a symmetric monotone metric is regular, if and only if λ^{-1} is integrable with respect to $h(\lambda)\,d\lambda$. We may obtain this by choosing for example

$$h_p(\lambda) = \begin{cases} 0, & \lambda \ < 1-p \\ p, & \lambda \ \ge 1-p \end{cases} \qquad 0 \le p \le 1$$

instead of the constant weight functions. Since

$$\int \frac{(\lambda^2-1)(1+t^2)}{(1+\lambda^2)(\lambda+t)(1+\lambda t)}\,d\lambda = \log\frac{1+\lambda^2}{(\lambda+t)(1+\lambda t)}$$

we are by tedious calculations able to obtain the expression

$$f_p(t) = \frac{1+t}{2}\left(\frac{4(1-p+t)(1+(1-p)t)}{(2-p)^2(1+t)^2}\right)^p \qquad t>0$$

for the normalized operator monotone functions represented by the $h_p(\lambda)$ weight functions [6, Theorem 1]. The corresponding Morozova-Chentsov functions are then given by

$$c_p(x,y) = \frac{(2-p)^{2p}}{(x+(1-p)y)^p((1-p)x+y)^p}\left(\frac{x+y}{2}\right)^{2p-1} \tag{20}$$

for $0 \le p \le 1$. The weight functions $h_p(\lambda)$ provides a continuously increasing bridge from the zero function to the unit function. But we cannot be sure that the corresponding Morozova-Chentsov functions are everywhere increasing, since we have adjusted the multiplicative constants such that all the functions $f_p(t)$ are normalized to $f_p(1) = 1$. However, since by calculation

$$\frac{\partial}{\partial p}f_p(t) = \frac{-2p^2(1-t)^2}{(2-p)^3(1+t)}\left(\frac{4(1-p+t)(1+(1-p)t)}{(2-p)^2(1+t)^2}\right)^{p-1} < 0,$$

we realize that the representing operator monotone functions are decreasing in p for every $t > 0$. In conclusion, we have shown that the symmetric monotone metrics given by (20) provides a continuously increasing bridge between the smallest and largest (symmetric and monotone) metrics, and that all the metrics in the bridge are regular except for $p = 1$.

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